

HISTORIES APPROACH TO QUANTUM MECHANICS

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I. Introduction

Quantum mechanics (QM), in its traditional formulation, is an eminently successful theory. It has, however, some unsatisfactory features. The experimentally verifiable predictions of the theory (which are generally probabilistic) about any quantum system are made with reference to an external observer who can test those predictions by repeatedly performing an experiment with prescribed initial conditions and invoking the frequency interpretation of probability. Such a formalism is clearly inadequate for the treatment of quantum dynamics of the universe as a whole.

Another unsatisfactory feature of the formalism is the postulate of state vector reduction in the theoretical treatment of quantum measurements. Its ad-hoc nature has always appeared unsatisfactory to those who care not only about the empirical adequacy of a theory but also about its intuitive appeal and conceptual coherence. A satisfactory formalism for QM should either provide an explanation of this reduction or else circumvent it.

The consistent history approach to QM is a reformulation/reinterpretation and extension of the traditional formalism of QM which is applicable to closed systems (in particular, the universe) and avoids the postulate of state vector reduction. The central objects in this approach are ‘histories’ which are temporal sequences of propositions about properties of a physical system. I shall try to cover the essential developments in this area allowing the choice of topics for the last one-third of the course to be somewhat subjective.

II. The Π Functions of Houtappel, Van Dam and Wigner; Primitive Concepts of Physical Theories

Objects analogous to what are now called histories were used way back in mid-sixties by Houtappel, Van Dam and Wigner (HVW) [1] who employed, in an article entitled ‘The conceptual basis and use of geometric invariance principles’, conditional probabilities $\Pi(B|A)$ where

$A = (\alpha, r_\alpha; \beta, r_\beta; \dots; \epsilon, r_\epsilon)$ denotes a set of measurements $\alpha, \beta, \dots, \epsilon$ (at times $t_\alpha, t_\beta, \dots, t_\epsilon$) with outcomes $r_\alpha, r_\beta, \dots, r_\epsilon$ and similarly $B = (\zeta, r_\zeta; \dots; \nu, r_\nu)$. The quantity $\Pi(B|A)$ denotes the probability of realization of the sequence B conditional to the realization of A. [I have changed HVW's notation $\Pi(A|B)$ to $\Pi(B|A)$ to bring it in line with the standard usage in probability theory.] The ordering of times $t_\alpha, \dots, t_\epsilon, t_\zeta, \dots, t_\nu$ is arbitrary. The objects $\Pi(B|A)$ are quite general and can be employed in classical as well as quantum mechanics. The measurements refer to external observers.

The central idea behind the HVW paper was ‘... to give a formulation of invariances directly in terms of the primitive concepts of physical theory, i.e. in terms of observations and their results.’ A more systematic identification of primitive elements of physical theories was done in ref [2]; they are :

- (i) Observations/measurements
- (ii) Evolution of systems (this involves the concept of time—discrete or continuous)
- (iii) Conditional predictions about systems: given some information about a system, to make predictions/retrodictions about its behaviour.

These three elements must be present (explicitly or implicitly) in every physical theory — classical, quantum or more general.

Note. It is important to note here that, when basics are formalised in this manner, the concept of *information* enters quite naturally at a very early stage in the foundations of physics.

The concept of *state* serves to integrate these three elements : a state is usually defined in terms of observable quantities; evolution of systems is generally described in terms of change of state with time and, in the formal statement of a problem of prediction/retrodiction, one generally gives information about a system by specifying an initial state.

The Π functions of HVW are objects defined in terms of primitive elements of physical theory. The same is also true of histories, as we shall see below. This, in the author's view, is the most important feature of history theories.

III. The Consistent History Formalism of Griffiths and Omnes [3-10]

3.1 Formula for the probability of a temporal sequence of measurement results in standard quantum theory [1,11]

We consider, for a typical quantum system S with initial state given by a density operator $\rho(t_0)$, a sequence of measurements at times t_1, t_2, \dots, t_n ($t_0 < t_1 < t_2 \dots < t_n$). Let the measurement results at these times be represented by projection operators P_1, P_2, \dots, P_n (which may refer to, for example, a particle's position in a certain domain, its momentum in a certain domain, its spin projection in a certain direction having a certain value, etc). We wish to calculate the joint probability

$$p = \text{Prob}(P_1, t_1; \dots; P_n, t_n | \rho(t_0)) \quad (1)$$

of these measurement results. Let

$$\rho(t_0) = \sum_{i=1}^k w_i |\phi_i\rangle\langle\phi_i| \quad w_i \geq 0; \quad \sum w_i = 1. \quad (2)$$

We shall first calculate this probability with $\rho(t_0) = |\phi\rangle\langle\phi|$ (pure state) and take (2) into consideration at the end.

The two main ingredients in this calculations are :

- (i) Between two consecutive measurements at times t_j, t_{j+1} , the system has unitary evolution given by the operator $U(t_{j+1}, t_j)$.
- (ii) At the completion of each measurement, we shall invoke the reduction postulate by applying the appropriate projection operator.

The (normalised) state just before the measurement at time t_1 is $|\psi(t_1)\rangle = U(t_1, t_0)|\phi\rangle$. Probability of a measurement result corresponding to $P_1 = 1$ is given by

$$p_1 = \text{tr}(P_1 |\psi(t_1)\rangle\langle\psi(t_1)|) = \langle\phi|U^\dagger(t_1, t_0)P_1U(t_1, t_0)|\phi\rangle = \langle\phi|P_1(t_1)|\phi\rangle \quad (3)$$

where we have introduced the Heisenberg picture projection operators

$$P_j(t_j) = U^\dagger(t_j, t_0)P_jU(t_j, t_0) \quad j = 1, 2, \dots, n. \quad (4)$$

Immediately after the measurement, the state is

$$D_1^{-1}P_1U(t_1, t_0)|\phi\rangle \quad (5)$$

where

$$D_1 = \langle\phi|U^\dagger(t_1, t_0)P_1U(t_1, t_0)|\phi\rangle^{1/2} = \langle\phi|P_1(t_1)|\phi\rangle^{1/2} = p_1^{1/2}. \quad (6)$$

The state just before the measurement at time t_2 is

$$|\psi(t_2)\rangle = D_1^{-1}U(t_2, t_1)P_1U(t_1, t_0)|\phi\rangle.$$

The probability of a measurement result $P_2 = 1$ at time t_2 (conditional on the measurement result $P_1 = 1$ at time t_1) is

$$\begin{aligned} p_2 &= \text{tr}(P_2|\psi(t_2)\rangle\langle\psi(t_2)|) \\ &= D_1^{-2} \langle\phi|U(t_1, t_0)^\dagger P_1 U(t_2, t_1)^\dagger P_2 U(t_2, t_1) P_1 U(t_1, t_0)|\phi\rangle \\ &= D_1^{-2} \langle\phi|P_1(t_1)P_2(t_2)P_1(t_1)|\phi\rangle. \end{aligned}$$

The joint probability of $P_1 = 1$ at time t_1 and $P_2 = 1$ at time t_2 is

$$\begin{aligned} p_1 p_2 &= \langle\phi|P_1(t_1)P_2(t_2)P_1(t_1)|\phi\rangle \\ &= \text{Tr}(P_1(t_1)P_2(t_2)P_1(t_1)|\phi\rangle\langle\phi|) \\ &= \text{Tr}(C_2|\phi\rangle\langle\phi|C_2^\dagger) \end{aligned} \tag{7}$$

where $C_2 = P_2(t_2)P_1(t_1)$. Denoting the temporal sequence $(P_1, t_1), \dots, (P_n, t_n)$ by α and defining

$$C_\alpha = P_n(t_n)P_{n-1}(t_{n-1})\dots P_1(t_1) \tag{8}$$

we have, by a continuation of the argument above, the joint probability of the outcomes α with the initial state $|\phi\rangle$ is (denoting by p_j the probability of $P_j = 1$ at time t_j conditional on $P_i = 1$ at times t_i for $i = 1, 2, \dots, j-1$)

$$p_1 p_2 \dots p_n = \text{Tr}(C_\alpha|\phi\rangle\langle\phi|C_\alpha^\dagger). \tag{9}$$

Finally, the desired probability (1) is

$$p = \text{Prob}(\alpha|\rho(t_0)) = \text{Tr}(C_\alpha\rho(t_0)C_\alpha^\dagger). \tag{10}$$

3.2 The decoherence functional in traditional quantum mechanics

Let α be the temporal sequence $(P_1, t_1; P_2, t_2; \dots; P_n, t_n)$ of the previous subsection and β the same as α except that, for a fixed t_j , P_j is replaced by another projector Q_j orthogonal to it ($P_j Q_j = Q_j P_j = 0$). Considering α and β as propositions [so that, for example, $\text{Prob}(\alpha|\rho(t_0))$ is the probability of the proposition α being true (with the initial state given to be $\rho(t_0)$)], let γ

be the proposition ‘ α or β ’; this is obtained by replacing P_j in α by $P_j + Q_j$. Clearly

$$C_\gamma = C_\alpha + C_\beta. \quad (11)$$

Eq(10) gives

$$\begin{aligned} Prob(\gamma|\rho(t_0)) &= Tr(C_\gamma\rho(t_0)) \\ &= Prob(\alpha|\rho(t_0)) + Prob(\beta|\rho(t_0)) + d(\alpha, \beta) + d(\beta, \alpha) \end{aligned} \quad (12)$$

where

$$d(\alpha, \beta) = Tr(C_\alpha\rho(t_0)C_\beta^\dagger). \quad (13)$$

Note that

$$d(\alpha, \beta) = d(\beta, \alpha)^* \quad (14)$$

$$d(\alpha, \alpha) \geq 0 \quad (15)$$

and

$$Prob(\alpha|\rho(t_0)) = d(\alpha, \alpha). \quad (16)$$

If α and β are to be mutually exclusive events in the sense of probability theory, we must have

$$Prob(\gamma|\rho(t_0)) = Prob(\alpha|\rho(t_0)) + Prob(\beta|\rho(t_0)). \quad (17)$$

The last two terms in eq(12) which prevent eq(17) from holding good, represent quantum interference effects. Eq(17) holds if and only if the following ‘decoherence condition’ is satisfied :

$$Red(\alpha, \beta) = 0. \quad (18)$$

The object $d(\alpha, \beta)$ is referred to as the *decoherence functional*. We will have much to do with this object later. Apart from the important equations (16) and (18), a significant feature of this object is that it contains information about the Hamiltonian H and the initial state $\rho(t_0)$. [If it is desired to make this explicit, we can replace the term $d(\alpha, \beta)$ by $d_{H, \rho(t_0)}(\alpha, \beta)$.] This fact becomes of considerable importance in quantum cosmology where, in the

context of quantum mechanics of the universe, the decoherence functional has information about the dynamics and the initial state of the universe.

Remark. Note that, in the traditional formalism of quantum mechanics, probabilities of histories can be calculated without bothering about decoherence conditions like (18). (This will not be true in the Griffiths' consistent history scheme presented in the next subsection.) These conditions are needed if we wish to construct a probability space of histories. The situation here is somewhat similar to the calculation of the probability of a sequence of coin-tossing results which can be done without formally defining a probability space whose elements are coin-tossing sequences.

3.3 Griffiths' consistent history scheme

In Griffiths' scheme, one retains the kinematical Hilbert space framework of traditional quantum mechanics and the quantum dynamical equations of Heisenberg and Schrödinger; the collapse postulate is discarded. It operates with closed quantum systems; measurement situations are accommodated by considering the system plus apparatus as a closed system. Quantum mechanics is treated as a statistical theory which assigns probabilities to families of temporal sequences of the form α, β considered above (but dropping the condition that actual measurements are performed at times t_1, t_2, \dots, t_n); they are referred to as *histories*.

The developments below refer to a closed quantum mechanical system S which has associated with it a Hilbert space \mathcal{H} . All operators and states are in the framework of this Hilbert space.

The basic interpretive unit is a *quantum event* consisting of a pair (t, P) where t is a point of time and P a projection operator describing a possible state of affairs (also referred to a quantum property) at time t . The negation of (the proposition corresponding to) the event $e = (t, P)$ (i.e. 'not e ') is the event $e' = (t, I - P)$. At any time t , one can define families of mutually exclusive and exhaustive sets of quantum events represented by mutually orthogonal projection operators constituting decompositions of unity :

$$P_j^{(a)} P_j^{(b)} = \delta_{ab} P_j^{(a)}, \quad \sum_a P_j^{(a)} = I \quad (19)$$

where the index j labels a family. A family typically corresponds to a definite observable (or, more generally, to a set of mutually commuting observables), different members of a family being projection operators corresponding to

non-overlapping domains in the (joint) spectrum of the observable(s), covering together the whole of the spectrum.

As indicated above, a *history* of the system S consists of a prescribed initial state $\rho(t_0)$ at some time t_0 and a time-ordered sequence of quantum events $(t_1, P_1), \dots, (t_n, P_n)$ where

$$t_0 < t_1 < \dots < t_n. \quad (20)$$

Note. Quite often, the initial state is understood (or prescribed separately) and history is defined as a time-ordered sequence of quantum events. This is convenient in, for example, quantum cosmology where the initial state of the universe is supposed to be fixed. We shall allow ourselves to be flexible in this matter.

To assign a probability to a history α , we must consider an exhaustive set of mutually exclusive histories (containing α) such that probabilities assigned to the histories in the set add up to unity. The standard way of constructing such a family is to fix a time sequence [say, the one in eq(20)] and the initial state $\rho(t_0)$ and construct, for each t_j , a decomposition of unity $P_j^{(\alpha_j)}$ satisfying eqs(19) [with $a = \alpha_j$ etc]; the desired family consists of all possible sequences

$$(t_1, P_1^{(\alpha_1)}), (t_2, P_2^{(\alpha_2)}), \dots, (t_n, P_n^{(\alpha_n)}). \quad (21)$$

Note that there is a freedom of choosing different partitions of unity at different times. The history represented by the sequence (21) is labelled by the symbol $\alpha = (\alpha_1, \dots, \alpha_n)$ which serves to distinguish different members of the family; the history itself is often referred to as α or by another symbol like h_α .

Some applications require more general histories (the so-called ‘branch-dependent’ histories) in which the choice of the projector at time t_j depends on the projectors at earlier times t_1, t_2, \dots, t_{j-1} . Such a history may be represented as a sequence of the form

$$(t_1, P_1^{(\alpha_1)}), (t_2, P_{(2, \alpha_1)}^{(\alpha_2)}), \dots, (t_n, P_{(n, \alpha_1, \dots, \alpha_{n-1})}^{(\alpha_n)}). \quad (22)$$

Unless stated otherwise, we shall work with histories of the form (21). We shall refer to the time sequence (t_1, \dots, t_n) as the *temporal support* (or simply *support*) of the history α .

Coarse-graining of histories is often employed in theoretical work. There are several ways of doing this. We mention here the simplest coarse-graining

in which the temporal supports are not changed (other methods will be described when needed); this is done by replacing one or more projectors $P_j^{(\alpha_j)}$ by the ‘coarser’ projectors

$$Q_j^{(\beta_j)} = \sum_{\alpha_j} b_{\alpha_j}^{\beta_j} P_j^{(\alpha_j)} \quad (b_{\alpha_j}^{\beta_j} = 0 \text{ or } 1) \quad (23)$$

Defining the Heisenberg projectors as in eq(4) (the time t_0 in that equation can, in fact, be chosen as some arbitrary reference time t_r), we define the *chain operator* for a history α [see eq(8)] as

$$C_\alpha = P_n^{(\alpha_n)}(t_n) \dots P_2^{(\alpha_2)}(t_2) P_1^{(\alpha_1)}(t_1). \quad (24)$$

Relations (19) (with appropriate replacement of indices) imply

$$\sum_{\alpha} C_\alpha = I \quad (25)$$

which represents the completeness of the family $\mathcal{F} = \{\alpha\}$.

Given two histories α and β in the family \mathcal{F} , the decoherence functional $d(\alpha, \beta)$ is defined as in eq(13) [with C_α given by eq(24)]; it satisfies the conditions (14) and (15). It is a measure of the quantum interference between the two histories α and β . (This should be clear from section 3.2.) The two histories α and β are said to satisfy the *consistency condition* (or *decoherence condition*) if eq(18) holds. When this condition holds for every pair of histories in the family \mathcal{F} , it is called a *consistent family*. [The word *framework* has been coined by Griffiths[12] for a consistent family of histories.] Legitimate probability assignments can be made only for histories in a consistent family.

Given a consistent family \mathcal{F} , the probability of a history α in \mathcal{F} is *postulated* to be given by eq(16). Equations (25), (14) and (18) ensure that

$$\sum_{\alpha} Prob(\alpha) = \sum_{\alpha} d(\alpha, \alpha) = 1. \quad (26)$$

It was found in most applications that, whenever the decoherence condition (18) holds, the stronger condition

$$d(\alpha, \beta) = 0, \quad \alpha \neq \beta \quad (27)$$

also holds. Being convenient to apply, it is the condition (27) that is more often used. Eq(18) is referred to as the *weak decoherence condition* and

(27) as the *medium strong decoherence condition*. For the treatment of the so-called ‘strong decoherence condition’, the reader is referred to ref[13].

It must be noted, however, that, whereas the conditions like (27) or stronger ones are sufficient to ensure consistency, eq(18) is a necessary and sufficient condition for it. The latter must be used if some question of principle or finer point is to be settled.

Note. Adoption of the formula (16) (which was *derived* in the traditional formalism of quantum mechanics *by employing the reduction postulate* as a *postulate* in the history theory might make a critical reader suspicious. “What is the great point of doing all this if the reduction postulate is being allowed to enter ‘from the back door’ in the formalism like this ?” she/he might ask. The point, stated in one sentence, is that the procedure being adopted constitutes a valid probability assignment (and, relatedly, a legitimate realization/representation of the relevant Boolean lattice/logic) – no matter in what context the formula being adopted first appeared.

Let me explain/elaborate.

In probability theory, one has a probability space consisting of a triple (Ω, \mathcal{B}, P) where Ω is a nonempty set (sample space or the space of elementary events), \mathcal{B} (the space of events) a family of subsets of Ω closed under unions, intersections and complements and P a real-valued function on \mathcal{B} (assignment of probability to events) satisfying the conditions

- (i) $P(A) \geq 0$ for all $A \in \mathcal{B}$
- (ii) $P(\emptyset) = 0$ and $P(\Omega) = 1$
- (iii) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$. When elements of \mathcal{B} are interpreted as propositions (for example, in the context of coin tossing, the event ‘head’, when expressed as ‘The head appears on tossing the coin’ is a proposition), the operations of union, intersection and complementation in \mathcal{B} correspond, respectively, to disjunction ($A \cup B$ corresponds to ‘A or B’ usually written $A \vee B$), conjunction ($A \cap B$ corresponds to ‘A and B’ usually written $A \wedge B$) and negation (A^c corresponds to ‘not A’ generally written as $\neg A$ or A^*); one then refers to \mathcal{B} as a Boolean logic or a Boolean lattice.

All probabilistic reasoning (whether applied to systems on which repeatable experiments can be performed or to closed systems like the universe) must be based on bonafide representations/realizations of Boolean logic. (In logic, such a realization is referred to a ‘universe of discourse’.) In the usual applications of probability theory, this requirement is (generally implicitly) taken care of.

The justification of the above-mentioned assignment of probabilities to

consistent histories is that a consistent family constitutes a bonafide representation of Boolean logic. For details, see Omnes [4,6,9].

3.4 Refinement and compatibility; Complementarity

We shall use the letters $\mathcal{F}, \mathcal{G}, \dots$ to denote consistent history families (CHF's). A CHF \mathcal{G} is said to be a *refinement* of another CHF \mathcal{F} (or \mathcal{F} is a *coarsening* of \mathcal{G}) if

- (i) $\text{supp}(\mathcal{G})$ contains $\text{supp}(\mathcal{F})$ (i.e. the time-points on which \mathcal{G} is defined include those on which \mathcal{F} is defined);
- (ii) at each time in $\text{supp}(\mathcal{F})$, the decomposition of unity employed in \mathcal{G} is either the same as that employed in \mathcal{F} or finer.

Two CHF's \mathcal{F} and \mathcal{F}' are said to be *compatible* if there is a CHF \mathcal{G} which is a refinement of both \mathcal{F} and \mathcal{F}' ; otherwise they are said to be *incompatible*.

More than one CHF's can describe the same physical process without being mutually compatible. (We shall see an example below.) This phenomenon is called *complementarity* and the mutually incompatible histories involved in the description are said to be complementary. This is a concrete formulation of Bohr's principle of complementarity in the consistent history framework. Its origin lies, of course, in the noncommutativity of the operator algebra in quantum kinematics (and the consequent possibility of constructing mutually noncommuting complete sets of commuting observables). For more details, see [9].

3.5 Example[9]

We consider a free spin 1/2 particle with the Hamiltonian $H = 0$ (which means that the translational degrees of freedom are suppressed). Let $t_0 < t_1 < t_2$. The initial state at time t_0 is the $\vec{S} \cdot \hat{n}_0 = 1/2$ state corresponding to $\rho(t_0) = \frac{1}{2}(I + \vec{\sigma} \cdot \hat{n}_0)$. At time t_2 , the spin component $\vec{S} \cdot \hat{n}$ is measured with the result $+1/2$. We wish to study as to what can be said about $\vec{S} \cdot \hat{n}'$ at an intermediate time t_1 .

We employ the following decompositions of identity at the times t_1 and t_2 :

$$\begin{aligned} t_1 : I &= P'_+ + P'_-; & P'_\pm &= \frac{1}{2}(I \pm \vec{\sigma} \cdot \hat{n}') \\ t_2 : I &= P_+ + P_-; & P_\pm &= \frac{1}{2}(I \pm \vec{\sigma} \cdot \hat{n}) \end{aligned}$$

We consider two histories α and β given by

$$\begin{aligned}\alpha &: (t_1, P'_+), (t_2, P_+) \\ \beta &: (t_1, P'_-), (t_2, P_+).\end{aligned}$$

We have (recalling that $H = 0$)

$$\begin{aligned}C_\alpha &= P_+(t_2)P'_+(t_1) = P_+P'_+ \\ C_\beta &= P_+(t_2)P'_-(t_1) = P_+P'_-.\end{aligned}$$

The consistency condition (18) gives

$$\text{Re}[\text{Tr}(C_\alpha \rho(t_0) C_\beta^\dagger)] = 0$$

which, after a straightforward calculation, gives the condition

$$(\hat{n} \times \hat{n}') \cdot (\hat{n}_0 \times \hat{n}') = 0 \quad (28)$$

which is satisfied by $\hat{n}' = \pm \hat{n}_0, \pm \hat{n}$ and some other values.

It is of interest to consider a couple of special cases :

- (i) $\hat{n} = \hat{n}_0$. Eq(28) now gives $\hat{n}' \times \hat{n}_0 = 0$ implying $\hat{n}' = \pm \hat{n}_0$. In this case, probabilistic reasoning can give information only about the spin component $\vec{S} \cdot \hat{n}_0$ at time t_1 . Eq(16) gives (with $\hat{n}' = \hat{n}_0$) $\text{Prob}(\alpha) = 1$ and $\text{Prob}(\beta) = 0$ which is consistent with the prediction of the traditional formulation of quantum mechanics. (In an eigenstate of an observable A with eigenvalue λ , a measurement of A always gives the value λ .)
- (ii) $\hat{n}_0 = \hat{i}, \hat{n} = \hat{k}$. Eq (28) now gives $n'_x n'_z = 0$. This condition permits, among others, the following two families which are mutually incompatible (and, therefore, complementary):

$$\begin{aligned}(a) & (t_1, S_x = \pm 1/2), (t_2, S_z = 1/2) \\ (b) & (t_1, S_z = \pm 1/2), (t_2, S_z = 1/2).\end{aligned}$$

IV. Understanding the Quasiclassical Domain

4.1 The programme [9,14]

An important agenda item for any scheme of quantum mechanics of the universe is to explain the existence of the quasiclassical domain of everyday

experience as an emergent feature of the quantum mechanical formalism. The term ‘quasiclassical’ refers to the fact that the classical deterministic laws give only an approximate description of this domain. (For example, we need a nonzero Planck constant to develop the foundations of classical statistical mechanics properly.) We would like to understand how, in the course of the quantum evolution of the universe, certain subsystems— a large majority of the macroscopic systems— manage to have a near-deterministic evolution and to have a clearer picture (than the deterministic classical laws provide) of this near-deterministic evolution.

In this section we shall present the histories approach to this problem keeping close to the paper [15] of Gell-Mann and Hartle who build up and improve upon the works of some earlier authors [16,17,5,6,9].

The arena of classical mechanics of a subsystem S of the universe is the phase space Γ of S . With a nonzero Planck constant, the uncertainty principle is operative which prohibits the classical evolution of S in terms of phase space trajectories. Moreover, quantum mechanics is probabilistic whereas classical mechanics is deterministic. In the consistent histories approach to quantum dynamics of the universe, what one hopes to show is that some appropriately chosen histories have, with probability very near unity,

- (i) their ‘events’ at various times described by cells of near- minimal size in Γ and
- (ii) the sequential locations of these cells are correlated by the classical laws of evolution.

The description of quantum mechanics of the universe at the deepest level is expected to include interacting quantum fields, strings or more fundamental objects and a specification of the initial state of the universe. (Recall that the definition of decoherence functional in the quantum mechanics of a closed system involves, besides a choice of time points, decompositions of unity at the chosen time points and the law of evolution, the specification of an initial state.) The existence of the quasiclassical domain depends crucially on the fundamental interactions operative at deeper levels as well as the initial state of the universe. If we had a reasonably complete theory of the quantum dynamics of the universe, that theory would dictate a natural choice of various ingredients for the histories to be chosen for the description of the quasiclassical domain. In the absence of such a theory, we shall be guided mainly by the intuition based on experience with classical physics.

We shall assume a fixed space-time background ignoring the problems relating to the description of quantum dynamics of the geometry of space-

time. We shall also assume a fixed time variable t . (This only means that we are operating in a fixed reference frame – Galilean or Lorentzian – and does not necessarily mean commitment to nonrelativistic mechanics.) The choice of time points in the histories to be considered will, therefore, be the usual one adopted in the previous section.

To decide on the decompositions of unity to be employed at the various time points, we must choose a (not necessarily complete) set of commuting observables. Typical classical observables of interest are pointer position on a measuring device, centre of mass of a cricket ball, etc. These are examples of the so-called *collective observables*[5,6,9]. Classically, they are described in terms of phase space variables. The events of interest (for the construction of the relevant histories) are represented classically by domains in the appropriate phase space which are neither too small nor too large. Construction of the corresponding quantum mechanical projection operators (or more generally, the so-called ‘quasi-projectors’) is a nontrivial task. One approach to constructing such operators is to employ integrals of coherent state projectors $|f_{qp}\rangle\langle f_{qp}|$ over appropriate domains D :

$$F = (2\pi\hbar)^{-1} \int_D |f_{qp}\rangle\langle f_{qp}| dq dp. \quad (29)$$

For a detailed treatment of these matters, we refer to Omnes[5,6,9].

We shall model reality by introducing a configuration space for the closed universe coordinatized by the configuration variables q^α ; these will be divided into the relevant variable x^a and another set Q^A which parametrise the configuration space of the ‘rest of the universe’ (which acts essentially as a heat bath) and are to be ignored. The systems with configuration variables x and Q will be referred to as A and B respectively. The histories considered will employ decompositions of unity corresponding to non-overlapping exhaustive sets of domains of the configuration space parametrised by the variables x^a .

Interaction between the distinguished and the ignored variables causes decoherence of the histories (chosen as above) by the rapid dispersal of the quantum mechanical phase information among the ignored variables [18,19]. (Details of this ‘environmental decoherence’ will be taken up elsewhere.) These interactions, apart from producing the expected classical behaviour (of the system with configuration variables x^a) characterized by predictability, also produce noise. Suppression of the effect of this noise to achieve classical predictability is achieved by additional coarse-graining of histories [15].

We shall not consider situations involving chaos in nonlinear classical systems; for a treatment of this, see ref[20].

4.2 Path integral representation of the decoherence functional[15,16,21,17,22]

We shall now obtain a path integral representation for the decoherence functional $d(\alpha, \beta)$ of eq(13) where

$$C_\alpha = P_n^{(\alpha_n)}(t_n) \dots P_1^{(\alpha_1)}(t_1)$$

and similarly for C_β . The projection operators $P_j^{(\alpha_j)}$ and $P_j^{(\beta_j)}$ represent the restriction of the configuration variables q^γ to the domains $\Delta_j^{(\alpha_j)}$ and $\Delta_j^{(\beta_j)}$ respectively at the indicated times.

Suppressing the superscript of the configuration variables, let q_S be the coordinate position operator in the Schrödinger picture and

$$q_H(t) = e^{iH(t-t_0)} q_S e^{-iH(t-t_0)}$$

the same in the Heisenberg picture. Their (generalized) eigenvectors satisfy, in obvious notation, the relations

$$\begin{aligned} q_S |q' \rangle &= q' |q' \rangle & q_H(t) |q', t \rangle &= q' |q', t \rangle \\ & & |q', t \rangle &= e^{iH(t-t_0)} |q' \rangle \\ \int dq' |q' \rangle \langle q'| &= I = \int dq' |q', t \rangle \langle q', t|. \end{aligned}$$

The path integral representation of the Schrödinger kernel is given by

$$\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')/\hbar} | q' \rangle = \int_{t', q'}^{t'', q''} Dq e^{iS[q]/\hbar}$$

where the integral Dq is over (an appropriate class of) configuration space trajectories $q(t)$ satisfying the end-point conditions $q(t') = q'$ and $q(t'' = q'')$. We shall write simply ρ for $\rho(t_0)$.

Putting $t_n = T$, we have
 $d(\alpha, \beta) = Tr(C_\alpha \rho C_\beta^\dagger)$

$$= \int \int \int \int dq'_f dq_f dq'_0 dq_0 \delta(q_f - q'_f) \langle q'_f, T | C_\alpha | q'_0, 0 \rangle \langle q'_0, 0 | \rho | q, 0 \rangle \langle q_0, 0 | C_\beta^\dagger | q_f, T \rangle. (30)$$

Now (taking, temporarily, $n = 2$)
 $\langle q'_f, T | C_\alpha | q'_0, 0 \rangle$

$$\begin{aligned}
&= \int \int dq_1 dq_2 \langle q'_f | P_2^{(\alpha_2)} | q_2 \rangle \langle q_2 | e^{-iH(t_2-t_1)/\hbar} | q_1 \rangle \langle q_1 | P_1^{(\alpha_1)} | q'_0 \rangle \\
&= \int_{[q'_0 \alpha q'_f]} Dq' e^{iS[q']/\hbar}
\end{aligned} \tag{31}$$

where the notation in the subscript in the last line means that the integration is over configuration space trajectories from q'_0 to q'_f consistent with the constraints of the history α at its temporal support. Now we can remove the restriction $n = 2$; eq(31) continues to be valid.

Proceeding similarly for the C_β^\dagger term in eq(30), we have, finally,

$$\begin{aligned}
d(\alpha, \beta) &= \int \int \int dq'_f dq_f dq_0 dq'_0 \delta(q_f - q'_f) \int_{[q'_0 \alpha q'_f]} Dq' \int_{[q_0 \beta q_f]} Dq'' e^{i(S[q'] - S[q''])/\hbar} \rho(q'_0, q_0) \\
&\equiv \int_{(\alpha)} Dq' \int_{(\beta)} Dq'' e^{i(S[q'] - S[q''])/\hbar} \rho(q'_0, q_0).
\end{aligned} \tag{32}$$

Introducing the systems A and B as in the previous subsection and defining the reduced density operator for the system of interest A as $\tilde{\rho}_A = Tr_{B\rho}$, we have, in the coordinate representation

$$\tilde{\rho}_A(x'_0, x_0) = \int dQ_0 \rho(x'_0, Q_0; x_0, Q_0). \tag{33}$$

Putting

$$S[q] = S_A[x] + S_B[Q] + S_I[x, Q] \tag{34}$$

and $Dq' = Dx' DQ'$ etc in eq(32), we carry out the Q-integrations. Defining the *influence phase* W by the relation

$$\begin{aligned}
&e^{iW[x', x; x'_0, x_0]} \tilde{\rho}_A(x'_0, x_0) \\
&\equiv \int \int DQ' DQ \delta(Q'_f - Q_f) \exp\left\{\frac{i}{\hbar}(S_B[Q'] + S_I[x', Q'] - S_B[Q] - S_I[x, Q])\right\} \\
&\quad \rho(x'_0, Q'_0; x_0, Q_0)
\end{aligned} \tag{35}$$

(where the notation $W[., ., ., .]$ means that it is a functional of the first two arguments and an ordinary function of the last two) we have

$$d(\alpha, \beta) = \int_\alpha \int_\beta Dx' Dx \delta(x'_f - x_f) e^{i(S_A[x'] - S_A[x] + W[x', x; x'_0, x_0])/\hbar} \tilde{\rho}_A(x'_0, x_0). \tag{36}$$

The influence phase W (which is a slight generalization of the quantity with the same name in ref[16]) represents the influence of the heat bath B on the dynamics of the system A in terms of the variables of A .

Generalizations of eqs(32) and (36) for the case when the histories α and β involve restrictions to domains of generalized momenta as well may be found in ref[15].

4.3 Classical equations of motion in linear quantum systems

We shall consider the model of eq(34) supplemented with the following additional assumptions :

(i) The action S_A is of the form

$$S_A[x] = \int_0^T dt \left[\frac{1}{2} \dot{x}^T(t) M \dot{x}(t) - \frac{1}{2} x^T(t) K x(t) \right] \quad (37)$$

where the superscript T denotes matrix transpose.

(ii) The part S_I is local in time and linear in x :

$$S_I[x, Q] = \int_0^T dt x^T(t) f(Q(t)) \quad (38)$$

where $f(Q)$ is linear homogeneous in the Q s.

(iii) The initial density matrix factorizes :

$$\rho(x'_0, Q'_0; x_0, Q_0) = \rho_A(x'_0, x_0) \rho_B(Q'_0, Q_0). \quad (39)$$

Eqs(33) and (39) give

$$\tilde{\rho}_A(x'_0, x_0) = \rho_A(x'_0, x_0). \quad (40)$$

(iv) The initial density matrix of the system B is of the form

$$\rho_B(Q'_0, Q_0) = e^{-B(Q'_0, Q_0)} \quad (41)$$

where $B(.,.)$ is a bilinear form.

Under these conditions, W has no explicit dependence on x'_0 and x_0 and has the general form

$$\begin{aligned} W[x', x] &= \frac{1}{2} \int_0^T dt \int_0^t dt' [x'(t) - x(t)]^T [k(t, t') x'(t') + k^*(t, t') x(t')] \\ &= \frac{1}{2} \int_0^T dt \int_0^t dt' [x'(t) - x(t)]^T \{ k_R(t, t') [x'(t') + x(t)] + i k_I(t, t') [x'(t') - x(t)] \}. \end{aligned} \quad (42)$$

Here $k(.,.) = k_R(.,.) + ik_I(.,.)$ is a complex matrix kernel. Moreover [23]

$$ImW[x, x'] \geq 0. \quad (43)$$

Putting

$$X(t) = \frac{1}{2}[x'(t) + x(t)]; \quad \xi(t) = x'(t) - x(t) \quad (44)$$

we have, from eq(36),

$$d(\alpha, \beta) = \int_{\alpha} Dx' \int_{\beta} Dx d_1[X, \xi] \quad (45)$$

with

$$d_1[X, \xi] = \delta(\xi_f) e^{iA[X, \xi]/\hbar} \rho_A(X_0 + \frac{\xi_0}{2}, X_0 - \frac{\xi_0}{2}) \quad (46)$$

where

$$A[X, \xi] = S_A[X + \frac{\xi}{2}] - S_A[X - \frac{\xi}{2}] + W[X, \xi]. \quad (47)$$

After a few integrations by parts, we have

$$A[X, \xi] = -\xi_0^T M \dot{X}_0 + \int_0^T dt \xi^T(t) e(t, X) + \frac{i}{4} \int_0^T dt \int_0^T dt' \xi^T(t) k_I(t, t') \xi(t') \quad (48)$$

where

$$e(t, X) = -M \ddot{X}(t) - KX(t) + \int_0^t dt' k_R(t, t') X(t'). \quad (49)$$

Eq(43) implies that d_1 has a decreasing exponential $\exp[-ImW/\hbar]$. It can be shown [16] that, with appropriate choice of domains $\Delta_j^{(\alpha_j)}$ and $\Delta_j^{(\beta_j)}$, significant contributions to $d(\alpha, \beta)$ in eq(45) can come only from $\xi(t)$ near zero and for $\alpha = \beta$. This implies, to a good approximation, the medium decoherence condition

$$d(\alpha, \beta) \approx 0, \quad \alpha \neq \beta. \quad (50)$$

We can now define the probability of a history in the decoherent family :

$$\begin{aligned} p(\alpha) &= d(\alpha, \alpha) \\ &\cong \int_{\alpha} DX [det(k_I/4\pi)]^{-1/2} \cdot \\ &\quad \exp[-\frac{1}{\hbar} \int_0^T dt \int_0^T dt' e^T(t, X) (k_I)^{-1}(t, t') e(t', X)] w_A(X_0, M \dot{X}_0). \end{aligned} \quad (51)$$

Here $w_A(X, M\dot{X})$ is the Fourier transform of $\rho_A(X_0 + \frac{\xi_0}{2}, X_0 + \frac{\xi_0}{2})$ with respect to ξ_0 (Wigner function).

It is clear from eq(51) that the histories with the largest probabilities will be those for which

$$0 = e(t, X] = -M\ddot{X}(t) - KX(t) + \int dt' k_R(t, t')X(t'). \quad (52)$$

This equation differs from the relevant Euler-Lagrange equation from the assumed action by the last term which is nonlocal in time; it arises from the interaction of the system A with the bath B.

When the bath B is treated as a collection of simple harmonic oscillators in thermal equilibrium at some temperature T_B , explicit expressions for the influence phase can be obtained. Replacing discrete bath oscillators by a continuum of oscillators with a cutoff frequency Ω and going to the Fokker-Planck limit [17]

$$kT_B \gg \hbar\Omega \gg \hbar\omega_R \quad (53)$$

(where ω_R is the frequency of the distinguished oscillator renormalized by its interaction with the bath), eq(52) takes the familiar form

$$e(t, X] \simeq -M\ddot{X}(t) - KX(t) - 2M\gamma\dot{X}(t) = 0. \quad (54)$$

In eq(51), individual classical histories are distributed according to the probabilities of their initial conditions given by the Wigner function $w_A(X_0, M\dot{X}_0)$. The fact that the Wigner function is not, in general, non-negative need not cause concern because positive definiteness of the probability $p(\alpha)$ has already been guaranteed earlier.

Eq(51) also reflects deviations from classical predictability. These deviations arise due to noise – the source of which is the same as the one responsible for decoherence – interaction with the bath B. For detailed treatment of this and related features, see[15].

V. Generalized Histories-based Quantum Mechanics ; Quantum Mechanics of Space-time[14,24,25]

5.1 The scheme of generalized quantum mechanics

Application of the concept of history as a time-sequence of projection operators to quantum cosmology would involve facing the problem of time

which arises due to non-availability of a preferred family of space-like hypersurfaces in quantum gravity. To bypass this problem, Hartle introduced a generalization of the history version of quantum mechanics in which histories are taken as the fundamental entities (without any reference to time sequences or projectin operators) and decoherence functionals are introduced by taking their properties noted earlier as defining properties. Formally, the generalized quantum mechanics of a closed system is defined by the following three ingradients :

(1) Fine-grained histories : These are objects representing the most refined description of dynamical evolution of the closed system to which one can contemplate assigning probabilities. For probability assignments, one considers sets $\{f\}$ of fine-grained histories which are exclusive and exhaustive. Typical examples of fine-grained histories are the set of (continuous) particle paths in nonrelativistic quantum mechanics and the set of field configurations in space-time in quantum field theory.

(2) Allowed coarse-grainings : A concrete scheme of generalized quantum mechanics generally permits a restricted class of possible coarse-grainings of histories. For example, the generalized quantum mechanics of gauge theories will permit only gauge-invariant classes of histories as coarse-grained histories.

(3) Decoherent functionals : A decoherence functional is a complex-valued function defined on pairs of histories in an exhaustive coarse-grained set $\{\alpha\}$ (which, as a special case, may be the set $\{f\}$ of fine-grained histories) satisfying the following conditions [see eqs(14, 15,26)]

(i) Hermiticity :

$$d(\alpha, \beta)^* = d(\beta, \alpha); \quad (55)$$

(ii) positivity :

$$d(\alpha, \alpha) \geq 0; \quad (56)$$

(iii) normalization :

$$\sum_{\alpha, \beta} d(\alpha, \beta) = 1; \quad (57)$$

(iv) biadditivity : Given an (exclusive, exhaustive) family $\{\bar{\alpha}\}$ which is obtained by (further) coarse-graining of a (possibly coarse-grained) family $\{\alpha\}$,

we have

$$d(\bar{\alpha}, \bar{\beta}) = \sum_{\alpha \in \bar{\alpha}} \sum_{\beta \in \bar{\beta}} d(\alpha, \beta). \quad (58)$$

Probabilities are assigned only to histories in a family $\{\alpha\}$ satisfying (generally approximately) a decoherence condition (with respect to some fixed decoherent functional d). Typical decoherence conditions are the weak decoherence condition

$$\text{Re } d(\alpha, \beta) \approx 0, \quad \alpha \neq \beta \quad (59)$$

and the medium decoherence condition

$$d(\alpha, \beta) \approx 0, \quad \alpha \neq \beta. \quad (60)$$

Probability of a history α (in a decoherent family) is given by

$$p(\alpha) = d(\alpha, \alpha). \quad (61)$$

The assumed properties of decoherence functionals ensure that this probability assignment satisfies the usual probability sum rules.

5.2 Quantum mechanics of space-time

General relativity (GR) is a theory of gravity as well as of space-time geometry. It is supposed to be the unique low energy limit of any quantum theory of gravity [26,27]. In this subsection, our objective is to construct a generalized quantum mechanics of space-time which, in an appropriate limit, gives equations of GR (in a sense analogous to the way equations of classical mechanics were obtained from the history version of quantum mechanics in the previous section).

The low energy theory has, as fine-grained histories, 4-dimensional manifolds equipped with smooth metrics of Lorentzian signature satisfying the Einstein equation and the matter field configurations satisfying appropriate field equations. For our quantum theory, therefore, we take as fine-grained histories 4-manifolds with arbitrary (continuous but not necessarily differentiable) Lorentz-signatured metrics $g_{\mu\nu}(x)$ and arbitrary (continuous) matter field configurations $\phi(x)$.

Generalized quantum mechanics permits topology change in space-time during quantum evolution. However, for simplicity, we shall consider only

spacetimes with the fixed topology $I \times M^3$ where I is a finite interval on the real line and M^3 a closed (i.e. compact without boundary) 3-manifold (which correspond to spatially compact universes over finite cosmological time-intervals). Accordingly, the 4-manifolds of interest have two M^3 boundaries; we shall call them $\partial M'$ and $\partial M''$. The induced 3-metrics and 3-dimensional field configurations on these boundaries will be denoted as $(h'_{ij}(\mathbf{x}), \chi'(\mathbf{x}))$ and $(h''_{ij}(\mathbf{x}), \chi''(\mathbf{x}))$ respectively.

In a gauge in which $g_{00}(x) = -1$ and $g_{0j} = 0$ (the so - called Gaussian gauge), we can write the 4-dimensional line element as

$$ds^2 = -dt^2 + h_{ij}(\mathbf{x}, t) dx^i dx^j. \quad (62)$$

The functions $h_{ij}(\mathbf{x}, t)$ describe a family of 3-metrics parametrised by the proper time t . In this description, therefore, the fine-grained histories may be thought of as curves in a space (called *superspace*) a typical point of which jointly represents a 3-metric $h_{ij}(\mathbf{x})$ and a spatial matter field configuration $\chi(\mathbf{x})$.

Allowed coarse-grainings in the generalized quantum mechanics being developed are partitions of the fine-grained histories into (exclusive, exhaustive) diffeomorphism-invariant classes. The most general description of such a coarse-graining is given in terms of ranges of values of diffeomorphism-invariant functionals of the 4-geometry and matter field configurations.

Statements about the universe in terms of observable features like near-homogeneity and isotropy of 3-geometry at late enough times generally corresponds to partitioning of its histories into two diffeomorphism-invariant classes—one in which the statement is true and the other in which it is not. (These classes are diffeomorphism-invariant because the statement does not involve coordinates or any other diffeomorphism non-invariant quantity.)

The so-called class operators C_α and the decoherence functionals $d(\alpha, \beta)$ are constructed in ref [25] essentially along the lines of the path integral constructions in eqs (31) and (32) taking due care of the invariances of the gravitational action. We shall give only a very rough outline of the constructions.

The Hilbert space on which the action of the operators C_α is defined is formally the space $\mathcal{H}^{(h, \chi)}$ of square-integrable functionals of 3-metrics and matter field configurations on the boundary hypersurfaces $\partial M'$ and $\partial M''$ mentioned earlier. (There are two Hilbert spaces involved; they are supposedly isomorphic.)

The expression for the matrix elements of C_α between the ‘coordinate eigenstates’ (analogous to $|q' \rangle$ and $|q'' \rangle$) arrived at, in ref[23], after considerable groundwork relating to the Hamiltonian treatment of the relevant gauge-invariances (diffeomorphisms and their phase space extensions) and their quantum treatment along the traditional Faddeev-Popov (FP) lines, is

$$\langle h'', \chi'' | C_\alpha | h', \chi' \rangle = \int_\alpha Dg D\phi \delta[\Phi[g, \phi]] \Delta_\Phi[g, \phi] e^{iS[g, \phi]}. \quad (63)$$

Here the functional integrations are over the 4-dimensional metrics g and the field configurations ϕ interpolating between the the boundary data appearing on the left. The δ -functional represents the gauge condition employed and Δ_Φ the corresponding FP factor; $S[g, \phi]$ is the 4-dimensional gravitational plus matter field action.

The constructions in ref [25] employ a time-symmetric formulation of the history formalism [28] which involves both the initial and the final density operators. (The traditional formalism is recovered as a special case of this by replacing the final density operator by the identity operator.) The initial and final density operators are assumed to be of the form

$$\rho_{in} = \sum_i p'_i |\Psi_i \rangle \langle \Psi_i| \quad \rho_f = \sum_j p''_j |\Phi_j \rangle \langle \Phi_j|. \quad (64)$$

. The decoherence functional is proposed to be given by

$$\begin{aligned} d(\alpha, \beta) &= \mathcal{N} Tr(\rho_f C_\alpha \rho_{in} C_\beta^\dagger) \\ &= \mathcal{N} \sum_{i,j} p''_j \langle \Phi_j | C_\alpha | \Psi_i \rangle \langle \Phi_j | C_\beta | \Psi_i \rangle^* p'_i \end{aligned} \quad (65)$$

where \mathcal{N} is a constant ensuring correct normalisation of the decoherence functional. The Hilbert space matrix elements of the class operators appearing in eq(65) are related to the objects in eq(63) by the relation

$$\langle \Phi_j | C_\alpha | \Psi_i \rangle = \Phi_j[h'', \chi''] \circ \langle h'', \chi'' | C_\alpha | h', \chi' \rangle \circ \Psi_i[h', \chi'] \quad (66)$$

where \circ is a Hermitian inner product between functionals on the superspace (not necessarily a positive definite one). Eq(66) is analogous to the relation (valid in a Hilbert space of square-integrable functions)

$$\langle \phi | A | \psi \rangle = \int \phi(x)^* \langle x | A | y \rangle \psi(y) dx dy. \quad (67)$$

[Note that, writing $\langle x|A|y \rangle = f_y(x)$, the x-integration in eq(67) (and similarly the y-integration) involves a scalar product. These scalar products are generally positive definite in the traditional formalism of quantum mechanics. In the formalism under consideration, however, this condition of positive definiteness can be dispensed with because positive definiteness of the probabilities in the theory is guaranteed by the defining properties of decoherent functionals. For the scalar product \circ in eq(66), Hartle settles on the DeWitt inner product given on page(195) of ref[25].]

The wave functions Φ_j and Ψ_i employed above have no direct probability interpretation; they appear only as part of the specification of the decoherence functional (65) (which is to perform its unusual job of determining which families of coarse-grained histories are decoherent and to assign probabilities to histories in a decoherent family).

The formalism presented above is generally covariant and circumvents the problem of time in quantum gravity. We did not need any preferred sets of space-like hypersurfaces in our constructions. There is no notion of state on a spacelike hypersurface.

In traditional quantum mechanics (with a given background spacetime) we do have a notion of state on a spacelike hypersurface. The formalism presented above, therefore, has the obligation to show how such a notion of state is recovered in appropriate limit.

This part of the program has not been completed in ref [25] (nor in any later work as far as the author is aware). Some useful points in this connection have been made in ref [25]; we shall, however, skip them.

VI. Some Mathematical Developments in Histories-based Theories

In this section, we shall briefly describe some works aimed at an organized mathematical development of histories-based theories building up on the idea of Gell-Mann and Hartle of taking histories as the basic objects and enriching the formalism with additional concepts like a generalization of the concept of time sequences based on partial semigroups, a multi-time generalization of quantum logic (to describe histories as propositions) etc.

6.1 Generalization of the concept of time-sequences; partial semi-groups

In histories-based theories, the time-sequences employed in the descrip-

tion of histories like

$$\alpha = (\alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_n}) \quad t_1 < t_2 < \dots < t_n \quad (68)$$

(where α_{t_i} are Schrödinger picture projection operators) serve only for book-keeping; the properties of time t as a real variable are not used. The mathematical structure which correctly describes the book-keeping of this sort and also makes provision for useful generalizations of the concept of time-sequence is that of a partial semigroup[11].

A *partial semigroup* (psg) is a nonempty set \mathcal{K} (whose elements will be denoted as s, t, u, \dots) in which a binary operation \circ between certain pairs of elements is defined such that $(s \circ t) \circ u = s \circ (t \circ u)$ whenever both sides are defined. A homomorphism of a psg \mathcal{K} into another psg \mathcal{K}' is a mapping $\sigma : \mathcal{K} \rightarrow \mathcal{K}'$ such that, for all $s, t \in \mathcal{K}$ with $s \circ t$ defined, $\sigma(s) \circ \sigma(t)$ is also defined and

$$\sigma(s \circ t) = \sigma(s) \circ \sigma(t). \quad (69)$$

If σ is invertible, it is called an isomorphism (automorphism if $\mathcal{K}' = \mathcal{K}$).

The psg involved in the book-keeping of temporal supports of histories in traditional quantum mechanics is

$$\mathcal{K}_1 = \{\text{finite ordered subsets of } \mathbb{R}\}$$

whose general element is of the form

$$t = \{t_1, t_2, \dots, t_n\}; \quad t_1 < t_2 < \dots < t_n. \quad (70)$$

If $s = \{s_1, \dots, s_m\} \in \mathcal{K}_1$ such that $s_m < t_1$, then $s \circ t$ is defined and

$$s \circ t = \{s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n\}. \quad (71)$$

It is useful to adopt the convention [25] that

$$\{t_1\} \circ \{t_1\} = \{t_1\}. \quad (72)$$

With this convention, we have $s \circ t$ defined when $s_m \leq t_1$. Note that elements of \mathcal{K}_1 admit an irreducible decomposition of the form

$$t = \{t_1\} \circ \{t_2\} \circ \dots \circ \{t_n\}. \quad (73)$$

Elements $\{t_i\}$ which cannot be further decomposed are called *nuclear*.

Interesting examples of partial semigroups whose elements are defined in terms of light cones and which are useful in the construction of histories in curved spacetimes may be found in ref[11] and [2]. These types of constructions have the potential to contribute towards solving the problem of time in quantum gravity.

6.2 Quasitemporal structures [11,2]

Histories of the form (68) also constitute a psg; we shall call it \mathcal{K}_2 . For $\alpha = (\alpha_{s_1}, \dots, \alpha_{s_m})$ and $\beta = (\beta_{t_1}, \dots, \beta_{t_n})$ with $s_m < t_1$, $\alpha \circ \beta$ is defined and is given by

$$\alpha \circ \beta = (\alpha_{s_1}, \dots, \alpha_{s_m}, \beta_{t_1}, \dots, \beta_{t_n}). \quad (74)$$

There is a homomorphism σ from \mathcal{K}_2 onto \mathcal{K}_1 given by

$$\sigma(\alpha) = s, \quad \sigma(\beta) = t, \quad \sigma(\alpha \circ \beta) = s \circ t. \quad (75)$$

The triple $(\mathcal{K}_2, \mathcal{K}_1, \sigma)$ defines a *quasitemporal structure* (a pair of psgs with a homomorphism of one onto the other). This concept formalizes and generalizes the idea of histories as temporal sequences of ‘events’.

Taking a clue from the traditional proposition calculus[30,31] where single time propositions are the basic entities, Isham[11] suggested that histories must be treated as multitime or more general propositions. He evolved a scheme of *quasitemporal theories* in which the basic objects were a triple $(\mathcal{U}, \mathcal{T}, \sigma)$ defining a quasitemporal structure. The space \mathcal{U} was called the *space of history filters* and was assumed to be a meet semilattice with the operations of partial order \leq (coarse-graining) and a meet/and operation \wedge (simultaneous realization of two histories). The space \mathcal{T} was called the *space of temporal supports*. To accomodate the operation of negation of a history, he proposed that the space \mathcal{U} be embedded in a larger space Ω (denoted as \mathcal{UP} in ref[11,32]) called the *space of history propositions*. This larger space was envisaged as having a lattice structure. (Later, we shall see in section 6.4 that the proper mathematical structure for it is that of an orthoalgebra.) Decoherence functionals were defined as complex valued functions on the space $\Omega \times \Omega$ satisfying the usual four conditions of hermiticity, positivity, biadditivity and normalization.

6.3 The history projection operator(HPO) formalism[11,33]

The logical operations on the history propositions are very similar to those of the traditional quantum logic[30,31] in which the elementary propositions (quantum events) have a standard representation as projection operators on the quantum- mechanical Hilbert space \mathcal{H} of the system. It is natural to look for a similar representation for the history propositions. Now, the usual class operators C_α occurring in the description of histories are products of (Heisenberg picture) projection operators. The product of two or more (generally noncommuting) projection operators need not be a projection operator. It follows that the class operators are generally not projection operators.

It is useful to note in this connection that the tensor product $P \otimes Q$ of two projection operators P and Q on \mathcal{H} is a projection operator on $\mathcal{H} \otimes \mathcal{H}$. Indeed, given (bounded) operators A, B, \dots on \mathcal{H} , we have, on $\mathcal{H} \otimes \mathcal{H}$,

$$(A \otimes B)(C \otimes D) = AC \otimes BD; \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger.$$

This gives

$$(P \otimes Q)^2 = P^2 \otimes Q^2 = P \otimes Q; \quad (P \otimes Q)^\dagger = P^\dagger \otimes Q^\dagger = P \otimes Q.$$

It follows that histories of the form (68) can be represented as projection operators in the Hilbert space

$$\tilde{\mathcal{H}} = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \dots \mathcal{H}_{t_n} \quad (76)$$

(where \mathcal{H}_{t_i} are copies of the Hilbert space \mathcal{H}) in the form

$$\alpha = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}. \quad (77)$$

Not all projection operators on $\tilde{\mathcal{H}}$ are ‘homogeneous’ projection operators of the form (77). A sum of two or more mutually orthogonal projection operators is a projection operator. When such a sum is not expressible as a homogeneous projection operator, it is called an inhomogeneous projection operator and a history represented as such an operator is called an inhomogeneous history.

The representation (77) of histories works well with the operation of negation. Denoting the negation of a history proposition α by $\neg\alpha$, we have, for $\alpha = P \otimes Q$,

$$\begin{aligned} \neg\alpha &= \neg(P \otimes Q) = I \otimes I - P \otimes Q \\ &= (I - P) \otimes Q + P \otimes (I - Q) + (I - P) \otimes (I - Q) \\ &= (\neg P) \otimes Q + P \otimes (\neg Q) + (\neg P) \otimes (\neg Q). \end{aligned} \quad (78)$$

The right hand side is a sum of three mutually orthogonal homogeneous projection operators which represent the three mutually exclusive (and exhaustive) situations corresponding to the negation of α .

For some further developments in history theory employing the HPO formalism, see Griffiths [7].

6.4 The algebraic scheme of Isham and Linden

In ref[32], Isham and Linden proposed a scheme more general than the quasitemporal formalism of Isham. In this scheme, even the concept of a quasitemporal structure was dispensed with. (The main motive for such a generalization comes from quantum gravity where one has the prospect of a ‘timeless’ formulation of dynamics.) In this scheme, the basic ingredients are the space Ω (denoted as \mathcal{UP} in their work) of history propositions and the space \mathcal{D} of decoherence functionals. The space Ω is assumed to be equipped with a structure (essentially that of an orthoalgebra[34]) incorporating the following operations :

- (i) Partial order (\leq). (a) Given $\alpha, \beta \in \Omega$, $\alpha \leq \beta$ means that α implies β (or is finer than β) or, equivalently, that β is coarser than α (or a coarse-graining of α).
- (b) The space Ω has two distinguished elements: the *null history proposition* 0 (a proposition that is always false) and a *unit history proposition* 1 which is always true. For all $\alpha \in \Omega$, we have

$$0 \leq \alpha \leq 1. \quad (79)$$

- (ii) Disjointness (\perp). This operation represents mutual exclusion: $\alpha \perp \beta$ means that if one of α and β is realised, the other must be ruled out (they are *disjoint*).

- (iii) Disjoint join operation (\oplus). (a) Given $\alpha \perp \beta$ in Ω , $\alpha \oplus \beta$ means ‘ α or β ’. It is assumed to be commutative and associative :

$$\alpha \oplus \beta = \beta \oplus \alpha; \quad (\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$$

whenever the expressions involved in these equations are defined.

- (b) The operations \leq and \oplus are related by the requirement that $\alpha \leq \beta$ if and only if there exists an element $\gamma \in \Omega$ such that $\beta = \alpha \oplus \gamma$.

- (iii) negation (\neg). For every $\alpha \in \Omega$, there is a unique element $\neg\alpha$ in Ω (meaning ‘not α ’ or negation of α) such that $\alpha \oplus \neg\alpha = 1$. The negation

operation satisfies the condition $\neg(\neg\alpha) = \alpha$. (This follows from the defining conditions of $\neg\alpha$ and the uniqueness of $\neg\alpha$.)

Note. The conditions defining an orthoalgebra are much weaker than those defining an (orthocomplemented) lattice. On a lattice, we have two connectives (binary operations $\Omega \times \Omega \rightarrow \Omega$), \wedge (meet: $\alpha \wedge \beta$ means ‘ α and β ’) and \vee (join: $\alpha \vee \beta$ means ‘ α or β ’) which are defined for every pair of elements. In contrast, an orthoalgebra has only one partial binary operation \oplus .

The space \mathcal{D} is treated as in section 6.2. The additivity condition on a decoherence functional now takes the following form: Given $\alpha \perp \beta$, we have

$$d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma).$$

A set $\alpha^{(i)} (i=1, \dots, n)$ of history propositions is *exclusive* if, members of this set are pairwise disjoint; it is called *exhaustive* (or *complete*) if it is exclusive and $\alpha^{(1)} \oplus \dots \oplus \alpha^{(n)} = 1$. Consistency /decoherence conditions and probability assignments on complete sets of history propositions are done in terms of decoherence functionals as before.

6.5 An axiomatic scheme for quasitemporal histories-based theories; symmetries and conservation laws in histories-based theories

This section contains a brief description of some work by the author and Yogesh Joglekar [29,2]. The work presented in ref[29] is an axiomatic development of dynamics of systems in the framework of histories and contains history versions of classical and traditional quantum mechanics as special cases. This work was motivated by the observation that while histories of a system contain, in principle, information about its measurable properties and dynamics, the Isham-Linden formalism is not adequately equipped to bring it out in an autonomous framework. We developed a more elaborate formalism for quasitemporal theories to fulfil this need.

Before stating the (five) axioms, we give a few more definitions relating to parial semigroups (psgs). A *unit element* in a psg \mathcal{K} is an element e such that $e \circ s = s \circ e = s$ for all $s \in \mathcal{K}$. An *absorbing element* in \mathcal{K} is an element a such that $a \circ s = s \circ a = a$ for all $s \in \mathcal{K}$. A psg may or may not have a unit and/or absorbing element; when either of them exists, it is unique. In a psg, elements other than the unit and absorbing elements are called *typical*.

A psg is called *directed* if, for any two typical elements s and t in it, if $s \circ t$ is defined, then $t \circ s$ is not defined. The psgs \mathcal{K}_1 and \mathcal{K}_2 introduced

in section 6.2 are directed. Directedness of the two psgs defining a quasi-temporal structure reflects the presence, in the quasi-temporal structure, a direction of flow of ‘time’.

The set of nuclear elements in a psg \mathcal{K} will be denoted as $\mathcal{N}(\mathcal{K})$. Clearly, $\mathcal{N}(\mathcal{K}_1) = \mathbb{R}$, the set of real numbers. A psg is called *special* if its elements admit semi-infinite irreducible decompositions [modulo redundancies implied by the convention (72)].

Axiom A_1 (*Quasitemporal Structure Axiom*). Associated with a (closed) dynamical system S is a *history system* $(\mathcal{U}, \mathcal{T}, \sigma)$ defining a quasitemporal structure as described earlier. The psgs \mathcal{U} and \mathcal{T} are assumed to be special and to satisfy the relation $\sigma[\mathcal{N}(\mathcal{U})] = \mathcal{N}(\mathcal{T})$.

Elements of \mathcal{U} (history filters) are denoted as α, β, \dots and those of \mathcal{T} (temporal supports) as τ, τ', \dots . If $\alpha \circ \beta$ and $\tau \circ \tau'$ are defined, we write $\alpha \triangleleft \beta$ (which means α precedes β) and $\tau \triangleleft \tau'$.

A_2 (Causality axiom). If $\alpha, \beta, \dots, \gamma \in \mathcal{N}(\mathcal{U})$ are such that $\alpha \triangleleft \beta \triangleleft \dots \triangleleft \gamma$ with $\sigma(\alpha) = \sigma(\gamma)$, then we must have $\alpha = \beta = \dots = \gamma$.

In essence, this axiom forbids histories corresponding to ‘closed time loops’. (This is the most primitive version of causality.) From these two axioms one can prove [29] that the two psgs \mathcal{U} and \mathcal{T} are directed and some other useful results. It is interesting to note that (in the present quasitemporal setting) a primitive version of causality implies a direction of flow of ‘time’.

A_3 (Logic Structure Axiom). Every space $\mathcal{U}_\tau \equiv \sigma^{-1}(\tau)$ for a $\tau \in \mathcal{N}(\mathcal{T})$ has the structure of a logic as defined in Varadarajan’s book [31].

A logic is essentially an orthoalgebra with meet (\wedge) and join (\vee) operations defined in it. Ref[29] contains the full statement of the axiom A_3 detailing the logic structure. Note that isomorphism of \mathcal{U}_τ s (as logics) for different nuclear τ s is not assumed. This generality gives the formalism additional flexibility so as to make it applicable to systems whose empirical characteristics may change with time (for example, the universe).

Using the logic structure of \mathcal{U}_τ s, a larger space Ω — the space of history propositions — was explicitly constructed and shown to be an ortho-algebra as envisaged in the scheme of Isham and Linden. Its subspace $\tilde{\mathcal{U}}$ representing the ‘homogeneous histories’ (which is obtained from \mathcal{U} after removing some redundancies) was shown to be a meet-semilattice as envisaged in the quasitemporal scheme of Isham [11].

A_4 (Temporal Evolution Axiom). The temporal evolution of a system as in axiom A_1 is given, for each pair $\tau, \tau' \in \mathcal{N}(\mathcal{T})$ such that $\tau \triangleleft \tau'$, by a set of mappings $V(\tau, \tau')$ of \mathcal{U}_τ onto $\mathcal{U}_{\tau'}$ which are logic homomorphisms (not necessarily injective) and which satisfy the composition rule $V(\tau'', \tau').V(\tau', \tau) = V(\tau'', \tau)$ whenever $\tau \triangleleft \tau' \triangleleft \tau''$ and $\tau \triangleleft \tau''$. (Note that the relation \triangleleft is generally not transitive.)

The space \mathcal{D} of decoherence functionals in the Isham-Linden scheme is not properly integrated with the basic framework. It is, for example, not made clear as to what distinguishes different elements of this space and when is a particular element relevant. In traditional quantum mechanics, as we have seen, a decoherence functional is determined by the evolution operator and the initial state. In the Isham-Linden formalism, there is no explicitly defined concept of state or of evolution; information about both of these is contained in the decoherence functional. It is, however, not clear how to bring out and use this information.

In our formalism we have both the concepts defined. A state at ‘time’ $\tau \in \mathcal{N}(\mathcal{T})$ is essentially a generalized probability on \mathcal{U}_τ [29]. Axiom A_5 associates a decoherence functional with a given law of temporal evolution and a given initial state, stipulates the (weak) decoherence condition and gives the usual rule for probability assignment for consistent/decoherent histories (elements of Ω).

The formalism would be complete if an explicit expression for the decoherence functional (analogous to those in section 3 and 4) is given. We were able to do it only for the Hilbert space -based theories (which have the traditional quantum mechanics as a special case) and for classical mechanics.

In the second paper [2], a systematic treatment of symmetries and conservation laws in the formalism of ref [29] was given. Symmetries were defined in a straightforward manner as natural invariances (automorphisms) of the basic structure. The directedness of the psg \mathcal{T} leads to a classification of symmetries into orthochronous (those preserving the ‘temporal order’ of events) and non-orthochronous.

A straightforward criterion for physical equivalence of histories was given:

All histories related to each other through orthochronous symmetry operations are physically equivalent.

This criterion covers several different notions of physical equivalence of histories considered by Gell-Mann and Hartle [35] as special cases.

In familiar situations, a reciprocal relationship between traditional symmetries (Wigner symmetries in quantum mechanics and Borel-measurable

transformations of phase space in classical mechanics) and symmetries in our formalism was established.

In a somewhat restricted class of theories, definition of a conservation law was given in the history language which agrees with the standard ones in familiar situations. In a subclass of these theories, a Noether-type theorem (implying a connection between continuous symmetries of dynamics and conservation laws) was proved.

The formalism evolved was applied to histories (of particles, fields, or more general objects) in general curved spacetimes. Sharpening the definition of symmetry so as to include a continuity argument, it was shown that a symmetry in our formalism implies a conformal isometry of the space-time metric. This condition is satisfied by all known symmetries in nature.

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